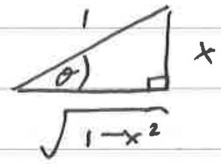
Mathematics: The core course for A-level (Bostock & Chandler)

Chapter 7: Trigonometric Identities

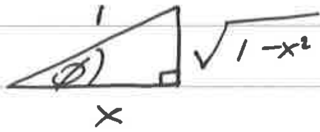
Exercise 7e

(1) $\arcsin x + \arccos x \equiv \frac{\pi}{2}$

let $\arcsin x = \theta$, $\therefore$ $x = \sin\theta$



let $\arccos x = \phi$, $\therefore$ $x = \cos\phi$



$\therefore$ $\theta + \phi \equiv \frac{\pi}{2}$ and $\therefore$ $\sin(\theta+\phi) \equiv \sin\frac{\pi}{2}$

Hence

$$\sin\theta\cos\phi + \sin\phi\cos\theta \equiv 1$$

$$\therefore \quad x \cdot x + \sqrt{1-x^2} \cdot \sqrt{1-x^2} \equiv 1$$

$$x^2 + \left(\sqrt{(1-x^2)}\right)^2 \equiv 1$$

$$x^2 + 1 - x^2 \equiv 1 \checkmark$$

(2) $\arctan \frac{1}{3} + \arctan \frac{1}{2} = \frac{\pi}{4}$

let $\arctan \frac{1}{3} = \theta$, $\therefore$ $\tan \theta = \frac{1}{3}$

let $\arctan \frac{1}{2} = \phi$, $\therefore$ $\tan \phi = \frac{1}{2}$

So $\theta + \phi = \frac{\pi}{4}$ and $\therefore$ $\tan(\theta + \phi) = 1$

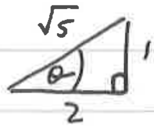
So $\dfrac{\tan \theta + \tan \phi}{1 - \tan\theta \tan\phi} = \dfrac{1/3 + 1/2}{1 - \frac{1}{3} \cdot \frac{1}{2}}$

$= \dfrac{5/6}{5/6} = 1$ ✓ $= RHS.$

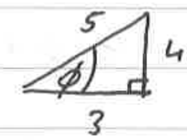
(3) $2 \arctan \frac{1}{2} = \arccos \frac{3}{5}$

So $2 \arctan \frac{1}{2} - \arccos \frac{3}{5} = 0$

let $\theta = \arctan \frac{1}{2}$, $\therefore$ $\tan \theta = \frac{1}{2}$



let $\phi = \arccos \frac{3}{5}$, $\therefore$ $\cos \phi = \frac{3}{5}$



So $2\theta - \phi = 0$, hence $\sin(2\theta - \phi) = 0$

$\therefore$ $\sin 2\theta \cos \phi - \sin \phi \cos 2\theta = 0$

$$2\sin\theta\cos\theta\cos\phi - \sin\phi\left(1-2\sin^2\theta\right)=0$$

From the diagrams above we have

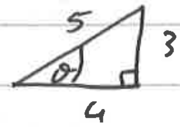
$$2\left(\frac{1}{\sqrt5}\right)\left(\frac{2}{\sqrt5}\right)\cdot\left(\frac{3}{5}\right) - \frac{4}{5}\left(1-2\left(\frac{1}{\sqrt5}\right)^2\right)=0$$

$$\frac{12}{25} - \frac{4}{5}\left(1-\frac{2}{5}\right) = 0 \quad \checkmark$$

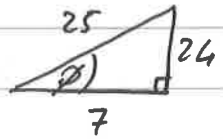
(4) $2\arctan\frac{3}{4} = \arccos\frac{7}{25}$

$$\therefore\ 2\arctan\frac{3}{4} - \arccos\frac{7}{25} = 0$$

let $\theta = \arctan\frac{3}{4}$, $\therefore\ \tan\theta = \frac{3}{4}$



let $\phi = \arccos\frac{7}{25}$, $\therefore\ \cos\phi = \frac{7}{25}$



So we have $2\theta - \phi = 0$

Therefore $\sin(2\theta-\phi) = 0$

Hence $\sin 2\theta\cos\phi - \sin\phi\cos 2\theta = 0$

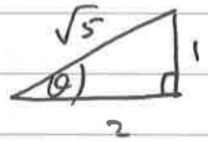
$$2\sin\theta\cos\theta\cos\phi - \sin\phi\left(1-2\sin^2\theta\right)=0$$

$$2\left(\frac{3}{5}\right)\cdot\left(\frac{4}{5}\right)\left(\frac{7}{25}\right) - \frac{24}{25}\left(1-2\left(\frac{3}{5}\right)^2\right)=0$$

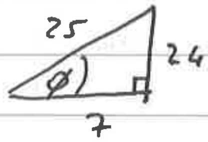
$$\frac{168}{625} - \frac{24}{25}\left(1 - \frac{18}{25}\right) = 0 \quad \checkmark$$

⑤ $4 \operatorname{arccot} 2 + \arctan \frac{24}{7} = \pi$

Let $\theta = \operatorname{arccot} 2$, $\therefore$ $\cot\theta = 2$



Let $\phi = \arctan \frac{24}{7}$, $\therefore$ $\tan\phi = \frac{24}{7}$



So $4\theta + \phi = \pi$, $\therefore$ $\cos(4\theta + \phi) = -1$

Hence $\cos 4\theta \cos\phi - \sin 4\theta \sin\phi =$

So $(1 - 2\sin^2 2\theta)\cos\phi - 2\sin 2\theta \cos 2\theta \sin\phi = -1$

$\therefore$ $\left(1 - 2\left(4\sin^2\theta \cos^2\theta\right)\right)\cos\phi$

$$- 2\left(2\sin\theta\cos\theta\right)\left(1 - 2\sin^2\theta\right)\sin\phi = -1$$

So $\left(1 - 2\left(4 \cdot \left(\frac{1}{\sqrt5}\right)^2 \cdot \left(\frac{2}{\sqrt5}\right)^2\right)\right) \cdot \frac{7}{25} - 2\left(2 \cdot \frac{1}{\sqrt5} \cdot \frac{2}{\sqrt5}\right)$

$$\times \left(1 - 2\left(\frac{1}{\sqrt5}\right)^2\right) \cdot \frac{24}{25} = -1$$

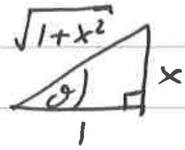
$\therefore$ $\left(1 - 8 \cdot \frac{4}{25}\right) \cdot \frac{7}{25} - \frac{8}{5} \cdot \left(1 - \frac{2}{5}\right) \cdot \frac{24}{25} = -1$

$$\therefore \quad \left(-\frac{7}{25}\right)\cdot\frac{7}{25} - \frac{8}{5}\left(\frac{3}{5}\right)\cdot\frac{24}{25} = -1$$

$$-\frac{49}{625} - \frac{576}{625} = -1 \quad \checkmark$$

(6) $\sin(2 \arctan x)$

Let $\arctan x = \theta$, $\therefore \tan\theta = x$



So $\sin(2\arctan x) = \sin(2\theta)$

$$= 2\sin\theta\cos\theta$$

$$= 2\,\frac{x}{\sqrt{1+x^2}}\cdot\frac{1}{\sqrt{1+x^2}}$$

$$= \frac{2x}{1+x^2}$$

(7) $\tan^{-1} x + \tan^{-1}\frac{1}{x}$

Let $\theta = \tan^{-1} x$, $\therefore \tan\theta = x$

Let $\phi = \tan^{-1}\frac{1}{x}$, $\therefore \tan\phi = \frac{1}{x}$

Hence $\tan^{-1} x + \tan^{-1}\frac{1}{x} = \theta + \phi$

$$\tan\left(\tan^{-1}x + \tan^{-1}\frac{1}{x}\right) = \tan(\theta+\phi)$$

$$= \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}$$

$$= \frac{x + \frac{1}{x}}{1 - x\cdot\frac{1}{x}} = \frac{x+\frac{1}{x}}{0}$$

So $\tan(\theta+\phi) = \text{"}\infty\text{"} \Rightarrow \theta + \phi = 90^\circ$

$\therefore \tan^{-1}x + \tan^{-1}\frac{1}{x} = 90^\circ$ or $\frac{\pi}{2}$

(8) $\tan\left(\arctan\frac{1}{3} + \arctan\frac{1}{4}\right)$

let $\arctan\frac{1}{3} = \theta$, $\therefore$ $\tan\theta = \frac{1}{3}$

let $\arctan\frac{1}{4} = \phi$, $\therefore$ $\tan\phi = \frac{1}{4}$

So $\tan(\theta+\phi) = \dfrac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}$

$$= \frac{1/3 + 1/4}{1 - \frac{1}{3}\cdot\frac{1}{4}} = \frac{7/12}{11/12} = \frac{7}{11}$$

(9) $\tan^{-1} 2 = \tan^{-1} 4 - \tan^{-1} x$

So $\tan^{-1} x = \tan^{-1} 4 - \tan^{-1} 2$

let $\theta = \tan^{-1} 4$, $\therefore$ $\tan\theta = 4$

let $\phi = \tan^{-1} 2$, $\therefore$ $\tan\phi = 2$

So $\tan^{-1} x = \theta - \phi$

Hence $\tan(\tan^{-1} x) = \tan(\theta - \phi)$

$$x = \frac{\tan\theta - \tan\phi}{1 + \tan\theta\tan\phi}$$

$$= \frac{4 - 2}{1 + 4\times 2} = \frac{2}{9}$$

(10) $\arctan(1+x) + \arctan(1-x) = \arctan 2$

let $\theta = \arctan(1+x)$, $\therefore$ $\tan\theta = 1+x$

let $\phi = \arctan(1-x)$, $\therefore$ $\tan\phi = 1-x$

$\therefore$ $\theta + \phi = \tan^{-1} 2$

So $\tan(\theta + \phi) = \tan(\arctan 2)$

$$\frac{\tan\theta + \tan\phi}{1 - \tan\theta\cdot\tan\phi} = 2 \Rightarrow \frac{(1+x) + (1-x)}{1 - (1+x)(1-x)} = 2$$

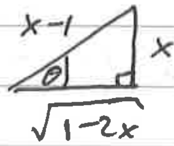
$$\therefore \quad \frac{2}{1-(1-x^2)} = 2$$

So $\quad \dfrac{2}{x^2} = 2 \quad \Rightarrow \quad x^2 = 1$

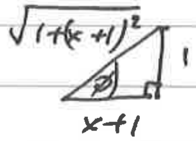
$\qquad\qquad\qquad\qquad \Rightarrow \quad x = \pm 1$

(11) $\sin^{-1}\left(\dfrac{x}{x-1}\right) + 2\tan^{-1}\left(\dfrac{1}{x+1}\right) = \dfrac{\pi}{2}$

Let $\theta = \sin^{-1} \dfrac{x}{x-1}$, $\therefore \quad \sin\theta = \dfrac{x}{x-1}$



Let $\phi = \tan^{-1} \dfrac{1}{x+1}$, $\therefore \quad \tan\phi = \dfrac{1}{x+1}$



So $\quad \theta + 2\phi = \dfrac{\pi}{2} \quad \Rightarrow \quad \sin(\theta + 2\phi) = \sin\dfrac{\pi}{2}$

So $\quad \sin\theta\cos 2\phi + \sin 2\phi \cos\theta = 1$

$\qquad \sin\theta\,(1 - 2\sin^2\phi) + 2\sin\phi\cos\phi\,\cos\theta = 1$

So $\quad \dfrac{x}{x-1}\cdot\left(1 - 2\left(\dfrac{1}{\sqrt{1+(x+1)^2}}\right)^2\right) + 2\left(\dfrac{1}{\sqrt{1+(x+1)^2}}\right)\left(\dfrac{x+1}{\sqrt{1+(x+1)^2}}\right)$

$$* \; \frac{\sqrt{1-2x}}{x-1} = 1$$

--- (to finish)